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Topological Radon Transforms and Projective Duality

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Abstract

We study various topological properties of projective duality in algebraic geometry by using the microlocal theory of sheaves developed by Kashiwara-Schapira [21]. In particular, in the real algebraic case we obtain some results similar to Ernström's ones [9] obtained in the complex case. For this purpose, we use constructible functions and their topological Radon transforms. We also generalize a class formula (i.e. a formula which expresses the degrees of dual varieties) in [10] to the case of associated varieties studied by Gelfand-Kapranov-Zelevinsky [12] etc. For the detail, see [26] and [27].

1 Introduction

We denote the projective space of dimension n over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$) by $\mathbb{P}_n$ and its dual space by $\mathbb{P}_n^*$. These spaces are naturally identified with the following sets.

$$\mathbb{P}_n = \{l \mid l \text{ is a line in } \mathbb{K}^{n+1} \text{ through the origin}\}, \tag{1.1}$$
$$\mathbb{P}_n^* = \{H' \mid H' \text{ is a hyperplane in } \mathbb{K}^{n+1} \text{ through the origin}\}. \tag{1.2}$$

Note that if we projectivize a hyperplane H' in $\mathbb{K}^{n+1}$ we obtain a hyperplane H in $\mathbb{P}_n$. Therefore in what follows we identify the dual projective space $\mathbb{P}_n^*$ with the set

$$\{H \mid H \text{ is a hyperplane in } \mathbb{P}_n\}. \tag{1.3}$$

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Definition 1.1 Let V be a projective variety in $\mathbb{P}_n$. We define the dual variety V^* of V by

$$V^* := \overline{\{H \in \mathbb{P}_n^* \mid \exists x \in V_{\text{reg}} \cap H \text{ s.t. } T_xV \subset T_xH\}} \quad (\subset \mathbb{P}_n^*). \tag{1.4}$$

When V is smooth, V^* is the set of hyperplanes tangent to V. As we see in the example below, even if V is smooth, V^* may be very singular in general.

Example 1.2 (i) Let $\iota_n : \mathbb{P}_1 \hookrightarrow \mathbb{P}_n$ be the Veronese embedding given by $[x : y] \mapsto [x^n : x^{n-1}y : \cdots : xy^{n-1} : y^n]$ and set $V = \iota_n(\mathbb{P}_1) \subset \mathbb{P}_n$. Then the dual $V^* \subset \mathbb{P}_n^*$ is a hypersurface defined by the classical discriminant for polynomials of degree n.

(ii) For $n \geq m$, consider the Segre embedding $\iota_{n,m} : \mathbb{P}_n \times \mathbb{P}_m \hookrightarrow \mathbb{P}_{(n+1)(m+1)-1}$ given by $([x_0 : \cdots : x_n], [y_0 : \cdots : y_n]) \mapsto [\cdots : x_iy_j : \cdots]$. Set $W = \iota_{n,m}(\mathbb{P}_n \times \mathbb{P}_m) \subset \mathbb{P}_{(n+1)(m+1)-1}$. Then the dual variety $W^* \subset \mathbb{P}^*_{(n+1)(m+1)-1}$ has very complicated singularities and the dual defect $\delta^*(W)$ of W (see (2.2) below) is $n - m$. Indeed, let $M_{(n+1),(m+1)}$ be the space of $(n+1) \times (m+1)$ matrices and identify the dual projective space $\mathbb{P}^*_{(n+1)(m+1)-1}$ with its projectivization $\mathbb{P}(M_{(n+1),(m+1)})$. Then the dual $W^* \subset \mathbb{P}^*_{(n+1)(m+1)-1}$ is explicitly written by

$$W^* = \mathbb{P}(\{A \in M_{(n+1),(m+1)} \mid \operatorname{rank} A \leq m\}). \tag{1.5}$$

Therefore the dual W^* admits a stratification defined by the ranks of matrices.

Many mathematicians were interested in the mysterious relations between projective varieties and their duals. Above all, they observed that the tangency of a hyperplane $H \in V^*$ with V is related to the singularity of the dual V^* at H. For example, consider the case of a plane curve $C \subset \mathbb{P}_2$ over $\mathbb{C}$. Then a tangent line l at an inflection point of C corresponds to a cusp of the dual curve C^*, and a bitangent (double tangent) line l of C corresponds to an ordinary double point of C^*. The most general results for complex plane curves were found in the 19th century by Klein, Plücker and Clebsch etc. (see for example, [34, Theorem 1.6] and [38, Chapter 7] etc.).

In the last two decades, this beautiful correspondence was extended to higher-dimensional complex projective varieties from the viewpoint of the geometry of hyperplane sections. In particular, after some important contributions by Viro [37] and Dimca [8] etc., Ernström proved the following remarkable result in 1994.

Theorem 1.3 [9, Corollary 3.9] *Let $V \subset \mathbb{P}_n$ be a smooth projective variety over $\mathbb{C}$. Take a generic hyperplane H in $\mathbb{P}_n$ such that $H \notin V^*$. Then for any hyperplane $L \in V^*$, we have*

$$\chi(V \cap L) - \chi(V \cap H) = (-1)^{n-1+\dim V - \dim V^*} \mathrm{Eu}_{V^*}(L), \tag{1.6}$$

where χ stands for the topological Euler characteristic and $\mathrm{Eu}_{V^*}\colon V^* \longrightarrow \mathbb{Z}$ is the Euler obstruction of V^* (introduced by Kashiwara [17] and MacPherson [24] independently).

Recall that the Euler obstruction Eu_{V^*} of V^* is a $\mathbb{Z}$-valued function on V^* which measures the singularity of V^* at each point of V^*. For example, Eu_{V^*} takes the value 1 on the regular part of V^*. Moreover, if we take a Whitney stratification $\bigsqcup_{\alpha \in A} V^*_\alpha$ of V^* consisting of connected strata, then Eu_{V^*} is constant on each stratum V^*_α. The values of Eu_{V^*} on a stratum V^*_α is determined by those on V^*_β's satisfying the condition $V^*_\alpha \subset \overline{V^*_\beta}$ (for more detail, see e.g. [18]).

Hence Ernström's result says that the jumping number of the topological Euler characteristics of hyperplane sections of V at L is expressed by $\mathrm{Eu}_{V^*}(L)$, that is, the singularity of the dual variety V^* at L.

The aim of this article is to introduce our results in the real algebraic case similar to this Ernström's one and to survey its theoretical background.

2 Main results

Consider a real projective space $X = \mathbb{RP}_n$ of dimension n and its dual $Y = \mathbb{RP}^*_n$. Let $M \subset X$ be a smooth real projective variety and $M^* \subset Y$ its dual variety.

We fix a μ-stratification $Y = \bigsqcup_{\alpha \in A} Y_\alpha$ of $Y = \mathbb{RP}^*_n$ consisting of connected strata and adapted to M^*. Note that Trotman [35] proved that this μ-condition is equivalent to famous Verdier's w-regularity condition.

Definition 2.1 We define a $\mathbb{Z}$-valued function $\varphi_M\colon Y \longrightarrow \mathbb{Z}$ on $Y = \mathbb{RP}^*_n$ by

$$\varphi_M(H) = \chi(M \cap H) \quad (H \in Y). \tag{2.1}$$

Since the function φ_M is defined by the topological Euler characteristics of hyperplane sections $M \cap H$ of M, to obtain results similar to Ernström's formula (1.6) it suffices to describe the function φ_M in terms of the singularities of M^*. We will show that the whole function φ_M can be reconstructed from one value $\varphi_M(y)$ at a point $y \in Y \setminus M^*$ and the singularities of M^*. First of all, for the above μ-stratification $Y = \bigsqcup_{\alpha \in A} Y_\alpha$ of $Y = \mathbb{RP}^*_n$ we can prove the following basic result.

Proposition 2.2 *The function φ_M is constant on each stratum Y_α.*

We denote the value of φ_M on Y_α by φ_α. Our main results are reconstruction theorems of φ_M. Namely, we can determine all the values φ_α's of φ_M from only one value $\varphi_M(y)$ at a point $y \in Y \setminus M^*$ and the topology of M^*.

To state the first theorem, we introduce two notations concerning dual varieties. Recall that the dual variety M^* is usually a hypersurface in $Y = \mathbb{RP}^*_n$.

Definition 2.3 (i) We denote the dual defect of M by

$$\delta^*(M) = (n-1) - \dim M^*. \tag{2.2}$$

(ii) For a conormal vector $\vec{p} \in T^*_{M^*_{\mathrm{reg}}} Y$ at $y \in M^*_{\mathrm{reg}}$, consider the second fundamental form

$$h_{M^*,\vec{p}} \colon T_y M^* \times T_y M^* \longrightarrow \mathbb{R}, \tag{2.3}$$

with respect to the canonical (Fubini-Study) metric of $Y = \mathbb{RP}^*_n$ and set

$$J_{\vec{p}} := \sharp\{\text{positive eigenvalues of } h_{M^*,\vec{p}}\} + \delta^*(M). \tag{2.4}$$

Now, let us state our first main theorem which describes the values of φ_M on $Y \setminus M^*_{\mathrm{sing}}$.

Theorem 2.4 ([26])

(i) *Assume that $\delta^*(M) > 0$. Then on $Y \setminus M^*$ the function φ_M is constant. Moreover for any $y \in M^*_{\mathrm{reg}}$ there exists an neighborhood U of y such that we have*

$$\varphi_M = d \cdot 1_Y + (-1)^{J_{\vec{p}}} 1_{M^*_{\mathrm{reg}}}. \tag{2.5}$$

on U, where d is the value of φ_M on $Y \setminus M^*$ and $\vec{p} \in T^*_{M^*_{\mathrm{reg}}} Y$ is a conormal vector at $y \in M^*_{\mathrm{reg}}$.

(ii) *Assume that $\delta^*(M) = 0$, that is M^* is a hypersurface in $Y = \mathbb{RP}_n^*$, and consider the following local situation. Let Y_{α_1} and Y_{α_2} be two strata in $Y \setminus M^*$, Y_β an open stratum in M^*_{reg} such that $Y_\beta \subset \overline{Y_{\alpha_i}}$ for $i = 1, 2$ and $\vec{p} \in T^*_{M^*_{\mathrm{reg}}} Y$ a conormal vector at a point $y \in Y_\beta$ pointing from Y_{α_1} to Y_{α_2} (see Figure 2.4.1 below). Then we have*

$$\varphi_{\alpha_2} - \varphi_{\alpha_1} = (-1)^{J_{\vec{p}}} - (-1)^{J_{-\vec{p}}} \quad (= 0, \pm 2), \tag{2.6}$$

$$\varphi_\beta = \begin{cases} \frac{1}{2}(\varphi_{\alpha_1} + \varphi_{\alpha_2}) & \text{if } \varphi_{\alpha_1} \neq \varphi_{\alpha_2}, \\ \varphi_{\alpha_1} + (-1)^{J_{\vec{p}}} & \text{if } \varphi_{\alpha_1} = \varphi_{\alpha_2}. \end{cases} \tag{2.7}$$

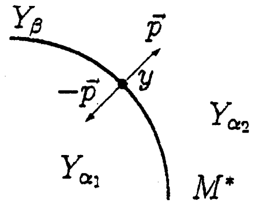

Figure 2.4.1

Remark 2.5 (i) If we rewrite (2.5) by Euler characteristics, we obtain an equality analogous to Ernström's formula (1.6). Namely, we have

$$\chi(M \cap L) - \chi(M \cap H) = (-1)^{J_{\vec{p}}} \tag{2.8}$$

for any $L \in M^*_{\mathrm{reg}}$, where $H \in Y \setminus M^*$ is a generic hyperplane in $Y = \mathbb{RP}_n^*$.

(ii) In the case where M^* is a hypersurface, the complement of M^* is divided into several connected components in general. So when we cross the hypersurface M^*, the value of the function φ_M may jump. Our formula (2.6) describes this jumping number in terms of the principal curvature $J_{\vec{p}}$ of M^*_{reg}.

Next, we state our second main theorem which reconstructs the values of φ_M on M^*_{sing}.

Theorem 2.6 ([26]) *Let $k \geq$ codim$_Y M^*$. Suppose that the values φ_α's on Y_α's satisfying* codim$_Y Y_\alpha \leq k$ *are already determined. Then the value φ_β on Y_β satisfying* codim$_Y Y_\beta = k + 1$ *is given by*

$$\varphi_\beta = \sum_{\alpha:\, Y_\alpha \cap B \neq \emptyset} \varphi_\alpha \cdot \{\chi(\overline{Y_\alpha} \cap B) - \chi(\partial Y_\alpha \cap B)\}. \tag{2.9}$$

Here we set $B = B(y, \varepsilon) \cap \{\psi < 0\}$ by taking a small enough open ball $B(y, \varepsilon)$ centered at a point $y \in Y_\beta$ and and a real-valued real analytic function ψ defined in a neighborhood of y satisfying $\psi^{-1}(0) \supset Y_\beta$ and

$$(y; \mathrm{grad}\psi(y)) \in T^*_{Y_\beta}Y \setminus \bigcup_{\alpha \neq \beta} \overline{T^*_{Y_\alpha}Y} \tag{2.10}$$

(see Figure 2.5.1 below).

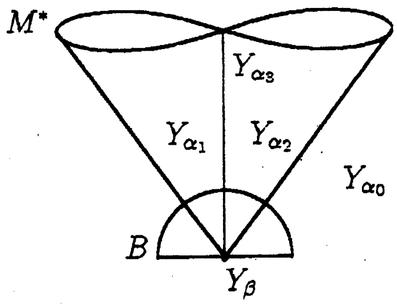

Figure 2.5.1

By Theorem 2.6, we can recursively determine the values φ_α's of φ_M by induction on the codimensions of strata Y_α's. Note that this representation of the function φ_M is completely analogous to that of the Euler obstructions in [18].

Example 2.7 Consider a smooth projective curve M defined by the homogeneous equation $x^4 + x^3z + z^4 - y^3z = 0$ in $\mathbb{RP}_2$ (see Figure 2.6.1 below).

Figure 2.6.1

Then the dual curve $M^* \subset \mathbb{RP}_2^*$ has a shape as in Figure 2.6.2 below. More precisely, as a μ-stratification of $Y = \mathbb{RP}_2^*$ adapted to M^*, we can take $Y = \bigsqcup_{i=0}^{11} Y_i$ in Figure 2.6.2. Since the last strata Y_{11} is contained in the line at infinity ($\simeq \mathbb{RP}_1$) of $\mathbb{RP}_2^*$ it does not appear in the figure.

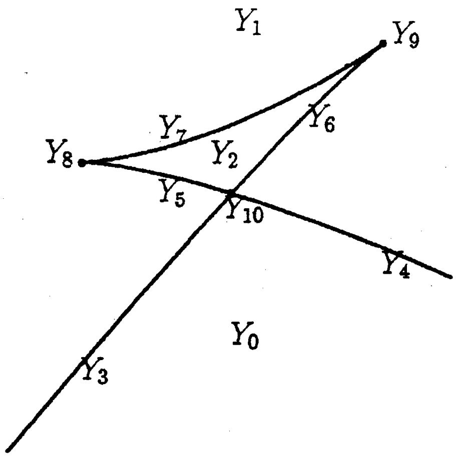

Figure 2.6.2

$\mathrm{codim}_Y Y_0 = 0$
$\mathrm{codim}_Y Y_1 = 0$
$\mathrm{codim}_Y Y_2 = 0$
$\mathrm{codim}_Y Y_3 = 1$
$\mathrm{codim}_Y Y_4 = 1$
$\mathrm{codim}_Y Y_5 = 1$
$\mathrm{codim}_Y Y_6 = 1$
$\mathrm{codim}_Y Y_7 = 1$
$\mathrm{codim}_Y Y_8 = 2$
$\mathrm{codim}_Y Y_9 = 2$
$\mathrm{codim}_Y Y_{10} = 2$

Now let us apply our two main theorems to this case. Denote by φ_i the value of the function φ_M on Y_i. Then we can easily see that φ_0 is 0. Starting from this value $\varphi_0 = 0$, we can recursively determine all the values φ_i's of φ_M as follows.

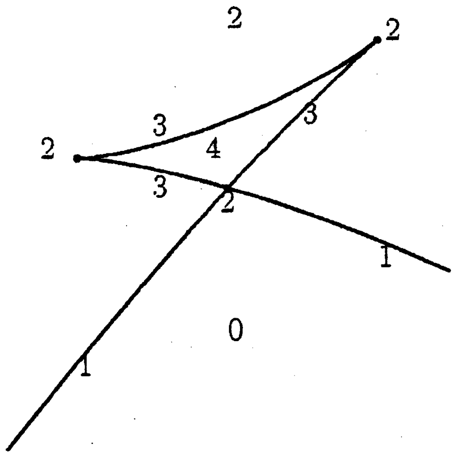

Figure 2.6.3

For example, by Theorem 2.4 the values φ_1 and φ_3 on Y_1 and Y_3 respectively can be calculated in the following way.

$$\begin{aligned} \varphi_1 &= \varphi_0 + (-1)^2 - (-1)^1 = 2, & (2.11)\\ \varphi_3 &= \frac{1}{2}(\varphi_0 + \varphi_1) = 1. & (2.12) \end{aligned}$$

Moreover, by Theorem 2.6 the value φ_{10} on Y_{10} is determined by φ_1, φ_2, φ_5 and φ_6 as follows.

$$\varphi_{10} = \varphi_1 \cdot 0 + \varphi_5 \cdot 1 + \varphi_2 \cdot (-1) + \varphi_6 \cdot 1 + \varphi_1 \cdot 0 = 2. \qquad (2.13)$$

In this case, we can easily check these results simply by counting the intersection numbers of M and lines in $\mathbb{RP}_2$. Namely, our results are the generalization of this very simple example to higher dimensional cases.

3 Theoretical background

Since the function φ_M in our main theorems is constant on each stratum of Y, we consider a class of such functions to study φ_M, which are called constructible functions.

Definition 3.1 Let X be a real analytic manifold. We say that a function $\varphi\colon X \longrightarrow \mathbb{Z}$ is constructible if there exists a locally finite family $\{X_i\}$ of compact subanalytic subsets X_i of X such that φ is expressed by

$$\varphi = \sum_i c_i \mathbf{1}_{X_i} \qquad (c_i \in \mathbb{Z}). \tag{3.1}$$

We denote the abelian group of constructible functions on X by $CF(X)$.

We define the operations of constructible functions in the following way.

Definition 3.2 ([21] and [37]) Let $f\colon Y \longrightarrow X$ be a morphism of real analytic manifolds.

(i) (The inverse image) For $\varphi \in CF(X)$, we define a function $f^*\varphi \in CF(Y)$ by

$$f^*\varphi(y) := \varphi(f(y)). \tag{3.2}$$

(ii) (The integral) Let $\varphi = \sum_i c_i \mathbf{1}_{X_i} \in CF(X)$ such that $\mathrm{supp}(\varphi)$ is compact. Then we define a topological (Euler) integral $\int_X \varphi \in \mathbb{Z}$ of φ by

$$\int_X \varphi := \sum_i c_i \cdot \chi(X_i). \tag{3.3}$$

(iii) (The direct image) Let $\psi \in CF(Y)$ such that $f|_{\mathrm{supp}(\psi)}\colon \mathrm{supp}(\psi) \longrightarrow X$ is proper. Then we define a function $\int_f \psi \in CF(X)$ by

$$\left(\int_f \psi\right)(x) := \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}). \tag{3.4}$$

From now on, we shall use various notions concerning derived categories of constructible sheaves. For the detail of these notions, see [21] etc. We denote by $\mathbf{D}^b(X)$ the derived category of bounded complexes sheaves of $\mathbb{C}_X$-modules on X. Its full subcategory consisting of complexes whose cohomology sheaves are $\mathbb{R}$-constructible is denoted by $\mathbf{D}^b_{\mathbb{R}-c}(X)$.

Recall also that the Grothendieck group $\mathbf{K}_{\mathbb{R}-c}(X)$ of $\mathbf{D}^b_{\mathbb{R}-c}(X)$ is a quotient group of the free abelian group generated by objects of $\mathbf{D}^b_{\mathbb{R}-c}(X)$ by the subgroup generated by

$$[F]-[F']-[F''] \quad (F' \longrightarrow F \longrightarrow F'' \xrightarrow{+1} \text{is a distinguished triangle}). \tag{3.5}$$

Then the natural morphism

$$\chi\colon \mathbf{K}_{\mathbb{R}-c}(X) \longrightarrow CF(X) \tag{3.6}$$

defined by $\chi([F])(x) = \sum_{j\in\mathbf{Z}}(-1)^j \dim H^j(F)_x$ $(x \in X)$ is an isomorphism. Moreover, by the isomorphism $\chi\colon \mathbf{K}_{\mathbb{R}-c}(X) \xrightarrow{\sim} CF(X)$ the operations of constructible functions that we introduced in Definition 3.2 correspond to those for $\mathbb{R}$-constructible sheaves. For example, let $f\colon Y \longrightarrow X$ be a morphism of real analytic manifolds and for $\psi \in CF(Y)$ take an object $G \in \mathbf{D}^b_{\mathbb{R}-c}(Y)$ such that $\psi = \chi(G)$. Then we have $\chi(Rf_*G) = \int_f \psi$ in $CF(X)$. In the same way, we can slightly generalize the notion of topological integrals of constructible functions as follows.

Definition 3.3 ([26]) Let U be a relatively compact subanalytic open subset of X and $\varphi \in CF(X)$. Take an object $F \in \mathbf{D}^b_{\mathbb{R}-c}(X)$ such that $\varphi = \chi(F)$ and set

$$\int_U \varphi := \chi(R\Gamma(U;F)). \tag{3.7}$$

We can easily check that the definition above does not depend on the choice of F such that $\varphi = \chi(F)$. Note that we do not have to assume that the support of φ is compact in U as in the usual definition (Definition 3.2 (ii)). Using this slight modification of the notion of topological integrals, we can express the R.H.S of (2.9) simply by $\int_B \varphi_M$. In fact, we used this fact in the proof of Theorem 2.6.

Now, let X be a real analytic manifold and denote by $\mathscr{L}_X$ the sheaf of conic ($\mathbb{R}_{>0}$-invariant) subanalytic Lagrangian cycles in the cotangent bundle T^*X of X. Its global section $H^0(T^*X;\mathscr{L}_X)$ is the abelian group of conic subanalytic Lagrangian cycles in T^*X. In 1985, Kashiwara [19] constructed a group homomorphism $CC\colon \mathbf{K}_{\mathbb{R}-c}(X) \longrightarrow H^0(T^*X; \mathscr{L}_X)$ and associated with each $[F] \in \mathbf{K}_{\mathbb{R}-c}(X)$ a Lagrangian cycle $CC([F])$ in T^*X. This Lagrangian cycle $CC([F])$ is called the characteristic cycle of $[F] \in \mathbf{K}_{\mathbb{R}-c}(X)$. The following very important theorem was proved also in [19] (see [21] for the detail).

Theorem 3.4 [21, Theorem 9.7.11] *There exists a commutative diagram*

$$\begin{tikzcd} & H^0(T^*X; \mathscr{L}_X) \arrow[dr, "\sim"] & \\ \mathbf{K}_{\mathbb{R}-c}(X) \arrow[ur, "\sim"', "CC"] \arrow[rr, "\sim", "\chi"'] & & CF(X) \end{tikzcd} \tag{3.8}$$

in which all arrows are isomorphisms.

By this theorem, we can reduce the problem of constructible functions (sheaves) to that of Lagrangian cycles.

4 Outline of the proof of main theorems

In this section, we give an outline of the proof of our main theorems. Let $X = \mathbb{RP}_n$ and $Y = \mathbb{RP}_n^*$ as before. Consider the incidence submanifold $S = \{(x, H) \in X \times Y \mid x \in H\}$ of $X \times Y$ and the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ & S \arrow[u, hook] \arrow[dl, "f"] \arrow[dr, "g"'] & \\ X & & Y, \end{tikzcd} \tag{4.1}$$

where p_1 and p_2 are natural projections and f and g are restrictions of p_1 and p_2 to X and Y respectively.

Definition 4.1 Let $\varphi \in CF(X)$. We define the topological Radon transform $\mathcal{R}_S(\varphi) \in CF(Y)$ of φ by

$$\mathcal{R}_S(\varphi) := \int_g f^*\varphi. \tag{4.2}$$

In particular, for a real analytic submanifold M of $X = \mathbb{RP}_n$ and a hyperplane H in $X = \mathbb{RP}_n$ ($\Longleftrightarrow$ $H \in Y = \mathbb{RP}_n^*$) we have

$$\mathcal{R}_S(1_M)(H) = \chi(M \cap H) \ \Big(= \varphi_M(H)\Big). \tag{4.3}$$

Therefore for the study of the function $\varphi_M \in CF(Y)$ it suffices to study the topological Radon transform $\mathcal{R}_S(\mathbf{1}_M)$. Using the isomorphisms in Theorem 3.4, instead of the topological Radon transform $\mathcal{R}_S\colon CF(X) \longrightarrow CF(Y)$ itself, we studied the corresponding operation for Lagrangian cycles (characteristic cycles). Then we found an isomorphism

$$\Psi\colon H^0(\dot{T}^*X; \mathscr{L}_X) \xrightarrow{\sim} H^0(\dot{T}^*Y; \mathscr{L}_Y), \tag{4.4}$$

where we set $\dot{T}^*X = T^*X \setminus T^*_X X$ and $\dot{T}^*Y = T^*Y \setminus T^*_Y Y$ (the zero-sections are removed). Moreover this operation Ψ is (up to some sign $\varepsilon = \pm 1$) the isomorphism of Lagrangian cycles induced by the canonical diffeomorphism $\Phi\colon \dot{T}^*X \xrightarrow{\sim} \dot{T}^*Y$ which coincides with the classical Legendre transform in the standard affine charts of $X = \mathbb{RP}_n$ and $Y = \mathbb{RP}_n^*$. Since the characteristic cycle $CC(\mathbf{1}_M)$ of $\mathbf{1}_M \in CF(X)$ is the conormal cycle $[\dot{T}^*_M X]$ in $\dot{T}^*X$, the characteristic cycle $CC(\mathcal{R}_S(\mathbf{1}_M))$ of the topological Radon transform $\mathcal{R}_S(\mathbf{1}_M) = \varphi_M$ is $\varepsilon[\Phi(\dot{T}^*_M X)]$. Set $\dot{\pi}_Y\colon \dot{T}^*Y \longrightarrow Y$ and $N = (\dot{\pi}_Y \circ \Phi)(\dot{T}^*_M X) \subset Y$. Then we can easily prove that N coincides with the dual variety M^* of M, which is a closed subanalytic subset of $Y = \mathbb{RP}_n^*$ (in classical terminology we call it a caustic or Legendre singularity). Moreover it turns out that the closure $\overline{\dot{T}^*_{N_{\mathrm{reg}}} Y}$ of the conormal bundle $\dot{T}^*_{N_{\mathrm{reg}}} Y$ in $\dot{T}^*Y$ is nothing but $\Phi(\dot{T}^*_M X)$ (see [16] for a similar argument). Then by using this very nice property of the characteristic cycle $CC(\varphi_M)$ we can reconstruct the function φ_M from the geometry of the dual variety $M^* = N$. Theorem 2.6 was proved in this way. To prove Theorem 2.4, we have to determine the sign $\varepsilon = \pm 1$, which is the most difficult part of our study. We could determine it by employing the theory of pure sheaves in [21]. More precisely, we expressed the Maslov indices of the Lagrangian submanifolds $\dot{T}^*_M X$ and $\dot{T}^*_{N_{\mathrm{reg}}} Y$ by the principal curvatures of M and N_{reg} respectively with the help of results in [11].

Remark 4.2 By the same argument as above, we can give a more transparent proof to the main results of Ernström [9] in the complex case.

5 Grassmann cases and class formulas

5.1 k-dual varieties

We shall generalize the situation considered in the previous sections to Grassmann cases and obtain similar results. Let $0 \leq k \leq n-1$ be an integer.

Recall that the Grassmann manifold consisting of k-dimensional planes in $\mathbb{P}_n$ is defined by

$$\begin{aligned} \mathbb{G}_{n,k} &= \{L' \mid L' \text{ is a } (k+1)\text{-dimensional linear subspace in } \mathbb{K}^{n+1}\} & (5.1)\\ &= \{L \mid L \text{ is a } k\text{-dimensional linear subspace in } \mathbb{P}_n\}. & (5.2) \end{aligned}$$

Note that $\mathbb{G}_{n,0} = \mathbb{P}_n$ and $\mathbb{G}_{n,n-1} = \mathbb{P}_n^*$. Then the notion of dual varieties is generalized to Grassmann cases as follows.

Definition 5.1 Let $V \subset \mathbb{P}_n$ be a projective variety. We define the k-dual variety $V^{\langle k \rangle}$ of V by

$$V^{\langle k \rangle} := \overline{\{L \in \mathbb{G}_{n,k} \mid \exists x \in V_{\mathrm{reg}} \cap L \text{ s.t. } V \not\pitchfork L \text{ at } x\}} \quad (\subset \mathbb{G}_{n,k}). \tag{5.3}$$

If $k = n-1$ the k-dual $V^{\langle k \rangle} \subset \mathbb{G}_{n,k} \simeq \mathbb{P}_n^*$ is nothing but the classical dual variety of V. In [12], Gelfand-Kapranov-Zelevinsky called $V^{\langle k \rangle}$ the associated variety of V and showed that $V^{\langle n-\dim V-1 \rangle}$ is a hypersurface.

5.2 Topological class formulas

From now on, we always assume that the ground field $\mathbb{K}$ is $\mathbb{C}$. Let $V \subset \mathbb{P}_n$ be a projective variety over $\mathbb{C}$ and $0 \leq k \leq n-1$ an integer. Assume that $V^{\langle k \rangle}$ is a hypersurface in $\mathbb{G}_{n,k}$.

Definition 5.2 [12, Proposition 2.1 of Chapter 3] Consider the Plücker embedding:

$$V^{\langle k \rangle} \subset \mathbb{G}_{n,k} \subset \mathbb{P}_{\binom{n+1}{k+1}-1}. \tag{5.4}$$

We call the degree of the defining polynomial of $V^{\langle k \rangle}$ in $\mathbb{P}_{\binom{n+1}{k+1}-1}$ the degree of $V^{\langle k \rangle}$ and denote it by $\deg V^{\langle k \rangle}$.

In [27], we proved the following topological class formula (i.e. a formula which expresses the degrees of dual varieties) for k-dual varieties by using Ernström's result [9] and some elementary formulas on constructible functions.

Theorem 5.3 ([27]) *In the situation as above, for generic linear subspaces $L_1 \simeq \mathbb{P}_{k-1}$, $L_2 \simeq \mathbb{P}_k$ and $L_3 \simeq \mathbb{P}_{k+1}$ of $\mathbb{P}_n$ we have*

$$\deg V^{\langle k \rangle} = (-1)^{(n-k)+\dim V+1} \left\{ \int_{L_1} \mathrm{Eu}_V - 2 \int_{L_2} \mathrm{Eu}_V + \int_{L_3} \mathrm{Eu}_V \right\}. \tag{5.5}$$

Corollary 5.4 *Let $L \simeq \mathbb{P}_{k+1}$ be a generic $(k+1)$-dimensional linear subspace of $\mathbb{P}_n$ and consider the usual dual variety $(V \cap L)^* \subset \mathbb{P}_{k+1}^*$ of $V \cap L \subset L \simeq \mathbb{P}_{k+1}$. Then we have*

$$\deg V^{\langle k \rangle} = \deg (V \cap L)^*. \tag{5.6}$$

The formula in Theorem 5.3 expresses the algebraic invariant $\deg V^{(k)}$ of $V^{(k)}$ by the topological data of V. In the case where $k = n-1$, we thus reobtain the topological class formulas obtained by Ernström [10], Parusinski and Kleiman [22] etc. See [34, Section 10.1] for an excellent review on this subject. In a forthcoming paper [28], from these topological class formulas we derive various more computable class formulas which extend the previous results obtained by Teissier and Kleiman [23] etc.

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